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A Sample Treatment of Langevin-Type Stochastic Differential Equations*

P. A. RUYMGAART

Department of Mathematics, University of Technology, Delft, The Netherlands

AND

T. T. SOONG

Faculty of Engineering and Applied Science, State University of New York, Buffalo, New York 14214

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A sample treatment of stochastic differential equations of the Langevin type is presented. It is shown that the formal mean square solutions of these equations are, in a certain sense, consistent with the solutions obtained based upon a sample theoretic approach. The procedure adopted in this development is to first construct an explicit sequence of Gaussian processes which approach in the limit the Brownian motion process. With derivatives of these processes as inputs to linear differential equations, we show that these equations possess solutions in the sample sense. These solutions are then shown to converge to the formal mean square solution of a differential equation having as input the white noise or the formal derivative of the Brownian motion process.

1. INTRODUCTION

Differential equations having random elements play a prominent role in the analysis of physical problems. To a large extent, problems of this type have been treated in the mean square sense. It is well recognized, however, that the sample solution properties associated with stochastic differential equations are of practical importance. A random process describing a physical phenomenon represents a family of ordinary functions. Each realization of a stochastic differential equation leads to a deterministic sample equation with a unique sample solution if the problem is well posed. Thus, to give physical meaning

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to mean square solutions, we need to consider sample equations and their associated sample solution trajectories.

Sample considerations are particularly pertinent in the case of Langevin-type differential equations where the non-homogeneous term is the white noise or the formal derivative of the Brownian motion process. As is well known, white noise does not exist in the sample sense nor in the mean square sense; and yet it has been one of the most widely used mathematical models for physical processes. We have seen the white noise used, for example, in the development of control theory, communication theory, and filtering and prediction problems.

The purpose of this paper is to show that the formal mean square solutions of Langevin-type differential equations are, in a certain sense, in agreement with the results based upon a sample theoretic approach. Brownian motion is treated as the limit of a sequence of stochastic processes having desirable properties. With derivatives of these processes as non-homogeneous terms, it is shown that the differential equations possess solutions in a sample sense. In turn, these solutions are shown to converge to the formal mean square solution.

It is noted that the procedure of approximating Brownian motion by a sequence of stochastic processes having "nice" sample properties has also been used by other investigators in similar situations. Wong and Zakai [1], for example, used this technique in a discussion of the Ito equation. Urbanik [2] has also used it in the study of stochastic functions as generalized functions. Some of the results in Section 3 are contained in the Theorem of Wong and Zakai; however, these results are obtained without using the involved Ito-calculus.

We begin by giving a general definition of *stochastic sample solutions* associated with stochastc differential equations. The basic probabilistic definitions used below can be found in Doob [3] or Skorokhod [4].

2. Stochastic Sample Solutions

Consider a system of random variables $\alpha_1, ..., \alpha_n$ and of sample continuous random functions $\phi_1(t), ..., \phi_m(t)$, $t \in [0, T]$, specified by means of the distribution function of $\{\alpha_1, ..., \alpha_n; \phi_1(t_1), ..., \phi_1(t_k); ...; \phi_m(t_1), ..., \phi_m(t_k)\}$ at all $\{t_1, ..., t_k\} \subset [0, T]$. Let $g_i(x_1, ..., x_n; f_1, ..., f_m; t)$, $i = 1, ..., n$, be a system of continuous mappings of $R^{n+m} \times [0, T]$ into R^1.

If $\xi_1(t), ..., \xi_n(t)$, $t \in [0, T]$, is a system of random functions possibly dependent upon the above random system, it follows from the continuity of g_i that at each $t \in [0, T]$

$$g_i(\xi_1(t), ..., \xi_n(t); \phi_1(t), ..., \phi_m(t); t), \qquad i = 1, ..., n, \tag{1}$$

is a system of Borel functions defined on the random variables $\xi_1(t),..., \xi_n(t)$; $\phi_1(t),..., \phi_m(t)$. Moreover, if $\xi_1(t),..., \xi_n(t)$ are also sample continuous, Equation (1) is a system of sample continuous random functions of t, $t \in [0, T]$.

Let us examine the following system of stochastic differential equations and initial conditions

$$\frac{d\xi_i(t)}{dt} = g_i(\xi_1(t),..., \xi_n(t); \phi_1(t),..., \phi_m(t); t), \qquad t \in [0, T]; \tag{2}$$

$$\xi_i(0) = \alpha_i, \qquad i = 1,..., n. \tag{3}$$

Here the "$=$" sign means that at fixed t the distribution function of the difference of left and right hand sides is the unit step function. The meaning of the symbol "d/dt" depends upon the type of solution desired. We use it here to mean continuous sample derivative.

DEFINITION. A stochastic sample solution of (2) is a system of random functions $\xi_i(t)$, $i = 1,..., n$, satisfying the following conditions:

(i) Almost all trajectories of the processes $\xi_i(t)$ are defined on an interval $[0, S] \subset [0, T]$, where $S > 0$ and independent of the trajectories.

(ii) At each fixed $t \in [0, S]$ the processes $\xi_i(t)$ are Borel functions defined on the random system $\{\alpha_1,..., \alpha_n; \phi_1(s),..., \phi_m(s), s \in [0, T]\}$.

(iii) The stochastic continuous sample derivatives of the processes $\xi_i(t)$ exist on $[0, S]$ and satisfy (2).

Then the stochastic sample solution is independent of the representations of the random system.

Under certain conditions, there is a unique stochastic sample solution to (2). Let $\{A_1(\omega),..., A_n(\omega); F_1(\omega, t),..., F_m(\omega, t), t \in [0, T]\}$ be a separable representation on a suitable probability space $\{\Omega, \mathcal{O\!l}, P\}$ of the above random system $\{\alpha_1,..., \alpha_n; \phi_1(t),..., \phi_m(t), t \in [0, T]\}$. We may assume that the exceptional sets of the resulting sample continuous representations $F_j(\omega, t)$, $j = 1,..., m$, are empty. Let us consider the system of differential equations and initial conditions

$$\begin{aligned} &\frac{dX_i}{dt} = g_i(X_1,..., X_n; F_1(\omega, t),..., F_m(\omega, t); t), \qquad t \in [0, T] \\ &X_i = A_i(\omega) \qquad \text{at} \qquad t = 0, \qquad i = 1,..., n. \end{aligned} \tag{4}$$

At each fixed ω, (4) represents a system of ordinary differential equations of real analisys. Since the functions g_i depend continuously on their variables and since at fixed ω the functions $F_j(\omega, t)$ depend continuously on t, the right hand sides of (4) are continuous functions of $X_1,..., X_n$ and

t with ω fixed. Moreover, if at each fixed ω the g_i satisfy a Lipschitz condition, it follows from the theorem of Cauchy that there exists a unique sample solution $X_i(\omega, t)$, $i = 1,..., n$ of (4) at each fixed ω. The t-domain of such a solution at ω is an interval $[0, S(\omega)] \subset [0, T]$ which depends on ω.

If condition (i) of the definition is satisfied, $X_i(\omega, t)$, $i = 1,..., n$ is a system of functions with the property that their sections at all $\omega \in \Omega$ satisfy (4) with $t \in [0, S]$. Again, according to the theorem of Cauchy, the functions $X_i(\omega, t)$ can be established as limits of the sequences $\{X_i^{(k)}(\omega, t)\}$ defined as follows

$$
\begin{aligned}
X_i^{(1)}(\omega, t) &= A_i(\omega); \\
X_i^{(2)}(\omega, t) &= A_i(\omega) + \int_0^t g_i(X_1^{(1)}(\omega, s),..., X_n^{(1)}(\omega, s); \\
&\qquad F_1(\omega, s),..., F_m(\omega, s); s)\, ds; \qquad (5) \\
&\cdots \\
X_i^{(k+1)}(\omega, t) &= A_i(\omega) + \int_0^t g_i(X_1^{(k)}(\omega, s),..., X_n^{(k)}(\omega, s); \\
&\qquad F_1(\omega, s),..., F_m(\omega, s); s)\, ds; \qquad t \in [0, S], \qquad i = 1, 2,..., n.
\end{aligned}
$$

Because of the continuity of all functions involved, the integrals in (5) exist as Riemann integrals. All members of the sequences (5) are Borel functions defined on $\{A_1(\omega),..., A_n(\omega); F_1(\omega, s),..., F_m(\omega, s); s \in [0, t]\}$ and so are the limits $X_i(\omega, t)$, $i = 1,..., n$. Hence, $X_i(\omega, t)$ represent random functions whose sample derivatives exist and satisfy condition (iii) of the definition.

Moreover, since the representation $X_i(\omega, t)$ are sample continuous, they are $\mathcal{A} \times \mathcal{B}$-measurable, where $\mathcal{B}$ is the Borel field of the t-segment $[0, S]$.

In closing, we remark that the techniques of the calculus in mean square are applicable if the stochastic sample solution is a second order process. If all functions contained in (1) are of second order and mean square continuous and if the functions g_i satisfy a suitable Lipschitz condition in mean square, the system (2) possesses a unique solution in the mean square sense provided that the derivatives are interpreted as mean square derivatives and that the initial values have finite second moments. This solution can also be established as a limit in mean square of the sequences (5) as the integrals in (5) can now be interpreted as mean square integrals. Hence, if both the sample conditions and the mean square conditions are satisfied, the sequences (5) converge a.s. as well as in mean square (and therefore in probability). It follows then that the system (2) possesses a unique stochastic sample solution and a unique mean square solution, and that both solutions are one and the same random function uniquely defined on the given random system.

3. The Linear Equation with Gaussian Input

In investigating the behavior of stochastic sample solutions of Langevin-type differential equations, we first need the solution of the equation where the non-homogeneous term is a Gaussian process, continuous in mean square and possessing sample continuous representations. We proceed in the following steps.

The Basic System

We define our basic system as follows:

(i) Given a set of deterministic real continuous functions, $a_0(t), a_1(t),\ldots, a_{n-1}(t)$, $t \in [0, \infty)$, and a (deterministic) differential equation

$$L\{x(t)\} = \frac{d^n x(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_0(t)\, x(t) = 0, \qquad t \in [0, \infty). \tag{6}$$

Let $\{x_1(t),\ldots, x_n(t)\}$ be its unique linearly independent set of (deterministic) solutions on $[0, \infty)$ with initial condition

$$\begin{bmatrix} x_1(0) & x_2(0) & \cdots & x_n(0) \\ x_1'(0) & x_2'(0) & \cdots & x_n'(0) \\ \vdots & & & \\ x_1^{(n-1)}(0) & x_2^{(n-1)}(0) & \cdots & x_n^{(n-1)}(0) \end{bmatrix} = I \tag{7}$$

It is assumed throughout that $x_i(t) \to 0$ as $t \to \infty$, $i = 1,\ldots, n$. Let $g(t, \tau)$, $(t, \tau) \in [0, \infty)^2$, be the unique (deterministic) one-sided Green's function associated with $L\{x(t)\} = 0$.

(ii) Given the finite dimensional joint distributions of a stochastic system consisting of a real stochastic process $\phi(t)$ on $[0, \infty)$ and real random variables $\alpha_0,\ldots, \alpha_{n-1}$. Let $\xi(t)$ be an unknown stochastic process satisfying the stochastic differential equation

$$L\{\xi(t)\} = \phi(t), \qquad t \in [0, \infty) \tag{8}$$

with random initial conditions

$$\xi(0) = \alpha_0,\ldots, \xi^{(n-1)}(0) = \alpha_{n-1}. \tag{9}$$

In the above, the equalities, such as $\xi(0) = \alpha_0$, stand for equalities with probability one. This convention is maintained throughout. Hence, $L\{\xi(t)\} = \phi(t)$, $t \in [0, \infty)$ means $P[L\{\xi(t)\} = \phi(t)] = 1$ at each fixed $t \geqslant 0$. The derivatives, such as $d\xi(t)/dt$, stand for stochastic sample derivatives unless stated otherwise. For clarity, the symbols $\overline{d}/dt$ and $\overline{\int_0^t}[\;]\,d\tau$ are occasionally used to denote derivatives and integrals in the mean square sense.

In order that $\xi(t)$ can be obtained in some stochastic sense, further conditions must be imposed on the stochastic process $\phi(t)$.

Case 1. Considering the basic system with the additional condition that $\phi(t)$ possesses sample continuous representations; the derivatives involved in the differential equation are to be interpreted as stochastic sample derivatives.

THEOREM 1. *The differential equation in Case 1 possesses a unique stochastic sample solution. It is asymptotically equal to*

$$\xi(t) = \int_0^t g(t, \tau)\,\phi(\tau)\,d\tau, \qquad t \geqslant 0. \tag{10}$$

The integral in (10) *is a stochastic sample integral.*

Proof. Let $\{A_0(\omega),\ldots, A_{n-1}(\omega); F(\omega, t)$ for all $t \in [0, \infty)\}$ be a representation on $\Omega \times [0, \infty)$ of $\{\alpha_0, \ldots, \alpha_{n-1}; \phi(t)$ on $[0, \infty)\}$ such that $F(\omega, t)$ is sample continuous. At each fixed $\omega \in \Omega$ we obtain a sample differential equation

$$L\{X_\omega(t)\} = F_\omega(t), \qquad t \in [0, \infty), \tag{11}$$

where $F_\omega(t)$ is continuous on $[0, \infty)$, with initial conditions

$$X_\omega(0) = A_0(\omega),\ldots, X_\omega^{(n-1)}(0) = A_{n-1}(\omega). \tag{12}$$

According to the deterministic theory of differential equations, the sample equation has a unique sample solution

$$X_\omega(t) = \sum_{i=1}^{n} A_{i-1}(\omega)\, x_i(t) + \int_0^t g(t, \tau)\, F_\omega(\tau)\, d\tau, \qquad t \geqslant 0. \tag{13}$$

Since the functions $x_i(t)$ and $g(t, \tau)$ are independent of ω, the totality of all sample solutions can be written in the form

$$X(\omega, t) = \sum_{i=1}^{n} A_{i-1}(\omega)\, x_i(t) + \int_0^t g(t, \tau)\, F(\omega, \tau)\, d\tau, \qquad (\omega, t) \in \Omega \times [0, \infty). \tag{14}$$

The integral above is the limit of a sequence of Riemann sums

$$\sum_k g(t, \tau_k')\, F(\omega, \tau_k')\,(\tau_k - \tau_{k-1}), \tag{15}$$

and, therefore, a Borel function defined on the $\mathcal{O}\!\mathcal{l}$-measurable sections $F_\tau(\omega)$ of $F(\omega, t)$ at values of τ in $[0, t]$. It follows that, since $X(\omega, t)$ is one of its representations, the process

$$\sum_{i=1}^{n} \alpha_{i-1} x_i(t) + \int_0^t g(t, \tau)\, \phi(\tau)\, d\tau, \qquad t \geqslant 0 \tag{16}$$

is the unique stochastic sample solution for this case. The integral in (16) is a stochastic sample integral and the solution is asymptotically equal to (10) as the functions $x_i(t)$ vanish as $t \to \infty$.

Using the property of the Green's function $g(t, \tau)$, it is easily shown that (16) possesses stochastic sample derivatives up to and including the nth order, that these derivatives possess sample continuous representations, and that their substitution into the differential equation leads to an equality with probability one at each fixed $t \geqslant 0$.

Case 2. Consider the basic system with the additional conditions that all given random quantities are of second order, and that $\phi(t)$ is mean square continuous on $[0, \infty)$; the derivatives contained in the differential equation are to be interpreted as mean square derivatives.

THEOREM 2. *The differential equation in Case 2 possesses a unique solution in the mean square sense. It is asymptotically equal to the second order process*

$$\xi(t) = \overline{\int_0^t} g(t, \tau)\, \phi(\tau)\, d\tau, \qquad t \geqslant 0. \tag{17}$$

The integral in (17) *is a mean square integral.*

Proof. It is well known in the mean square analysis (or analysis in Banach spaces) that in this case

$$\sum_{i=1}^{n} \alpha_{i-1} x_i(t) + \overline{\int_0^t} g(t, \tau)\, \phi(\tau)\, d\tau, \qquad t \geqslant 0, \tag{18}$$

is the unique solution in the mean square sense. It is asymptotically equal to (17) as $x_i(t) \to 0$ ($i = 1, \ldots, n$) as $t \to \infty$.

Given Case 2, it can also be shown that the deterministic function

$$E\xi(t) = \sum_{i=1}^{n} x_i(t)\, E\alpha_{i-1} + \int_0^t g(t, \tau)\, E\phi(\tau)\, d\tau, \qquad t \geqslant 0 \tag{19}$$

is the unique solution of the deterministic differential equation

$$L\{E\xi(t)\} = E\phi(t), \qquad t \geqslant 0 \tag{20}$$

with deterministic initial conditions

$$E\xi(0) = E\alpha_0, \ldots, E\xi^{(n-1)}(0) = E\alpha_{n-1}. \tag{21}$$

Let $\xi(t) = E\xi(t) + \eta(t)$. Substituting it into the differential equation gives

$$\eta(t) = \sum_{i=1}^{n} x_i(t)\{\alpha_{i-1} - E\alpha_{i-1}\} + \overline{\int_0^t} g(t, \tau)\{\phi(\tau) - E\phi(\tau)\}\, d\tau, \qquad t \geqslant 0 \tag{22}$$

with $E\eta(t) = 0$, as the unique mean square solution of the stochastic differential equation

$$L\{\eta(t)\} = \phi(t) - E\phi(t), \qquad t \geqslant 0 \tag{23}$$

with random initial conditions

$$\eta(0) = \alpha_0 - E\alpha_0, \ldots, \eta^{(n-1)}(0) = \alpha_{n-1} - E\alpha_{n-1}. \tag{24}$$

The derivatives given above are to be interpreted as mean square derivatives.

The preceding remarks show that the mean parts can be analyzed separately within the framework of deterministic theory. There is no loss of generality to consider only random quantities with zero expectations.

Case 3. Consider the linear system with the additional conditions that all given random quantities are jointly Gaussian with zero expectations, and that $\phi(t)$ is mean square continuous and possesses sample continuous representations.

It is seen that $\phi(t)$ is completely specified by its covariance function in this case. It is denoted by

$$\Gamma_\phi(s, t) = E\phi(s)\,\phi(t), \qquad (s, t) \in [0, \infty)^2. \tag{25}$$

THEOREM 3. *The differential equation in Case 3 possesses a unique stochastic sample solution and a unique mean square solution on* $[0, \infty)$. *These solutions, represented by* (16) *and* (18) *respectively, are one and the same stochastic process. The solution is asymptotically equal to*

$$\xi(t) = \int_0^t g(t, \tau)\,\phi(\tau)\, d\tau = \overline{\int_0^t} g(t, \tau)\,\phi(\tau)\, d\tau, \qquad t \geqslant 0. \tag{26}$$

(26) *is a Gaussian process with zero mean and covariance function*

$$\Gamma_\xi(s, t) = \int_0^s \int_0^t g(s, \sigma)\, g(t, \tau)\, \Gamma_\phi(\sigma, \tau)\, d\sigma\, d\tau, \qquad (s, t) \in [0, \infty)^2. \tag{27}$$

Proof. At fixed t the integrals in (16) and (18) can be established by means of the same sequence of Riemann sums. This sequence is convergent almost surely in Case 1. In Case 2 it is convergent in mean square, and hence in probability. The limits are thus equal with probability one.

Since the Riemann sums are linear combinations of random variables taken from a Gaussian process, their mean square limits at values $t \geqslant 0$ constitute a Gaussian process. Hence, the stochastic sample solution (26) is a Gaussian process.

4. The Linear Equation with White Noise Input

Formally, white noise is the derivative of Brownian motion process. As it exists neither as a stochastic sample derivative nor as a mean square derivative, the basic system defined in the previous section is meaningless if $\phi(t)$ is white noise. Moreover, Brownian motion itself, although possessing sample continuous representations, is in a strict sense not a suitable mathematical model of a moving particle. Over any interval of time, its trajectories are not differentiable and are not of bounded variation (i.e., having infinite length).

Let $\beta(t)$, $t \geqslant 0$, be the Brownian motion process with parameter σ^2. In the formal mean square treatment, white noise is substituted into the mean square solution (17) in Case 2, giving the formal solution

$$\text{“}\overline{\int_0^t} g(t, \tau) \frac{d\beta(\tau)}{d\tau}\, d\tau\text{”}. \tag{28}$$

This formal integral is then replaced by the Riemann-Stieltjes mean square integral

$$\overline{\int_0^t} g(t, \tau)\, d\beta(\tau).$$

This integral exists since the covariance function of $\beta(t)$ is of bounded variation on $[0, t]^2$. Upon partial integration and using the properties that $\beta(0) = 0$, $g(t, t) = 0$, and $g(t, \tau)$ is continuously differentiable with respect to τ, we obtain

$$\overline{\int_0^t} g(t, \tau)\, d\beta(\tau) = -\overline{\int_0^t} \beta(\tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau. \tag{29}$$

As a function of t, (29) is a Gaussian process with independent increments. It is mean square continuous and it possesses sample continuous representations.

However, we stress that stochastic sample solutions model the physical processes under investigation. If we replace $\phi(t)$ in Case 1 by the formal stochastic sample derivative "$d\beta(t)/dt$", the formal stochastic sample solution

$$\text{“}\int_0^t g(t, \tau) \frac{d\beta(\tau)}{d\tau} d\tau\text{”} \tag{30}$$

is obtained. Let us interpret this formal integral as the stochastic sample Riemann-Stieltjes integral

$$\int_0^t g(t, \tau)\, d\beta(\tau). \tag{31}$$

Although the trajectories of $\beta(\tau)$ are not of bounded variation on $[0, t]$, this integral still exists as a stochastic sample integral. This is justified from the fact that the integral

$$-\int_0^t \frac{\partial g(t, \tau)}{\partial \tau} \beta(\tau)\, d\tau$$

exists as such and is equal to the former by virtue of the theorem on partial integration. At a fixed t it is equal to (29) with probability one. Hence, in a certain sense, (29) can also be seen as a formal stochastic sample solution.

Based upon the physical considerations stated in the Introduction, it is constructive to (i) enhance physical plausibility to Brownian motion and its formal derivative as mathematical models and (ii) show that the formal solution (29) has meaning in the stochastic sample sense. We proceed by considering Brownian motion as the limit of a sequence of stochastic processes $\{\beta_n(t)\}$ whose stochastic sample derivatives $d\beta_n(t)/dt$ exist and satisfy the conditions imposed on $\phi(t)$ in Case 3 of the preceding section. To each n, Theorem 3 is applicable with $\phi(t) = d\beta_n(t)/dt$, resulting a corresponding stochastic sample solution $\xi_n(t)$. We then show that $\xi_n(t)$ converges to (29) as $n \to \infty$.

The power of this procedure rests upon the way in which the sequences $\{\beta_n(t)\}$ and $\{\xi_n(t)\}$ converge to $\beta(t)$ and (29), respectively. Loosely speaking, the stronger both sequences converge, the better the result (29) as a means of estimating the trajectories of the solution $\xi_n(t)$ when n is large. Consequently, we conclude this paper by

(A) defining a sequence of stochastic processes $\beta_n(t)$ having properties stated above,

(B) investigating the convergence of $\beta_n(t)$ to $\beta(t)$, and

(C) investigating the convergence of $\xi_n(t)$ to (29).

(A) *Definition of* $\beta_n(t)$.

Given the Brownian motion process $\beta(t)$, $t \in [0, T]$, with parameter σ^2, let $B(\omega, t)$, $(\omega, t) \in \Omega \times [0, T]$, be a sample continuous representation of $\beta(t)$, where Ω is the point set of a suitable probability space $\{\Omega, \mathcal{A}, P\}$. This representation is maintained throughout the remaining part of this paper. It is assumed that $T < \infty$.

Let us consider a sequence of partitions p_n of $[0, T]$ such that the mesh of p_n tends to zero as $n \to \infty$. Let the partition p_n be defined by the subdivision points $0 = t_0 < t_1 < \cdots < t_n = T$.

Let $h(x)$ be a real, monotone, continuous function, $0 \leqslant x \leqslant 1$, such that $h(0) = 0$ and $h(1) = 1$. Moreover, it is assumed that $h(x)$ is K times continuously differentiable on $[0, 1]$ and its first K derivatives vanish at $x = 0$ and $x = 1$. Let us denote

$$h_i(t) = h\left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right), \qquad i = 1, 2,..., n. \tag{32}$$

The sequence $\{\beta_n(t), t \in [0, T]\}$ is defined as follows: Given p_n,

$$\begin{gathered} \beta_n(t) = \beta(t_{i-1}) + h_i(t)\,[\beta(t_i) - \beta(t_{i-1})], \\ t_{i-1} \leqslant t \leqslant t_i, \qquad i = 1, 2,..., n. \end{gathered} \tag{33}$$

It is seen that $\beta_n(t)$ is a Borel function, defined on $\beta(t)$ by interpolation. We easily see that $\beta_n(t)$ has the following properties:

$\beta_n(t_i) = \beta(t_i)$, $i = 0, 1,..., n$.

$\beta_n(t)$, $t \in [0, T]$, possesses the finite degree of randomness, $n + 1$.

$\beta_n(t)$ is a Gaussian process; it is of second order with $E\beta_n(t) = 0$ on $[0, T]$ and $\beta_n(0) = 0$. The random variables $\beta_n(t_i) - \beta_n(t_{i-1})$, $i = 1,..., n$, are independent, but $\beta_n(t)$ is not a process with independent increments.

$\beta_n(t)$ is mean square continuous on $[0, T]$. It is even K times continuously mean square differentiable on $[0, T]$. Its covariance function is easily computed. It can be shown that, due to the monotony of $h(x)$, its total variation on $[0, T]^2$ is equal to $\sigma^2 T$, independent of n. It is thus equal to the total variation on $[0, T]^2$ of the covariance function of $\beta(t)$ itself.

$\beta_n(t)$ possesses the sample continuous representation

$$\begin{gathered} B_n(\omega, t) = B(\omega, t_{i-1}) + h_i(t)\,[B(\omega, t_i) - B(\omega, t_{i-1})], \\ t_{i-1} \leqslant t \leqslant t_i, \qquad i = 1, 2,..., n. \end{gathered} \tag{34}$$

Its trajectories are K times continuously sample differentiable. Hence, the stochastic sample derivatives of $\beta_n(t)$ up to and including the Kth order exist and possess sample continuous representations. At each fixed $t \in [0, T]$, all K stochastic sample derivatives can also be seen as mean square derivatives.

Finally, the stochastic sample derivative $d\beta_n(t)/dt$ satisfies the conditions imposed on $\phi(t)$ in Case 3.

(B) *The Convergence of* $\beta_n(t)$ *to* $\beta(t)$.

THEOREM 4. *The sequence* $\{\beta_n(t)\}$ *converges to* $\beta(t)$ *in mean square, uniformly in* $t \in [0, T]$.

Proof. Being mean square continuous on the compact set $[0, T]$, $\beta(t)$ is uniformly continuous in mean square on $[0, T]$. Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that $[s, t] \subset [0, T]$ and $|s - t| < \delta$ imply $\|\beta(s) - \beta(t)\| < \epsilon$, where $\|\beta(s) - \beta(t)\|$ is the norm of $\beta(s) - \beta(t)$ in the usual $\mathscr{L}_2$-space of second order random variables.

Partition p_n defines $\beta_n(t)$. Suppose that n is sufficiently large so that the mesh of p_n and the meshes of all following partitions are less than δ. Given $t \in [0, T]$, there is an i such that $t \in [t_{i-1}, t_i]$. Then $|t_i - t_{i-1}| < \delta$ and $|t - t_{i-1}| < \delta$. By definition (33) we have

$$\|\beta_n(t) - \beta(t_{i-1})\| = |h_i(t)| \cdot \|\beta(t_i) - \beta(t_{i-1})\| < \epsilon. \tag{35}$$

It then follows that

$$\|\beta_n(t) - \beta(t)\| \leqslant \|\beta_n(t) - \beta(t_{i-1})\| + \|\beta(t) - \beta(t_{i-1})\| < 2\epsilon \tag{36}$$

independent of $t \in [0, T]$. (The proof could also be established with the aid of the covariance functions of $\beta_n(t)$).

It is analogously proved that at each fixed ω the trajectories of $B_n(\omega, t)$, $n = 1, 2, \ldots$, converge to $B_\omega(t)$, uniformly in $t \in [0, T]$ as $n \to \infty$.

It should be pointed out that this convergence is not uniform in $(\omega, t) \in \Omega \times [0, T]$. However, it can be shown that, since all representations considered are finite and $\mathcal{O}\!\mathcal{l} \times \mathscr{B}$-measurable ($\mathscr{B}$ is the usual Borel field of the segment $[0, T]$), according to a theorem of Egorov [5], the convergence of $B_n(\omega, t)$ to $B(\omega, t)$ is uniform on $\Omega \times [0, T]$ outside of an $\mathcal{O}\!\mathcal{l} \times \mathscr{B}$-measurable subset of $\Omega \times [0, T]$ of arbitrarily small positive $P \times L$-measure (L is the usual Lebesque measure on $\mathscr{B}$). This subset is identified more precisely in the following Theorem.

THEOREM 5. *Given* $\epsilon > 0$, *there is a set* $A \in \mathcal{O}\!\mathcal{l}$ *with* $P(A) < \epsilon$ *such that* $B_n(\omega, t) \to B(\omega, t)$ *as* $n \to \infty$, *uniformly in* $(\omega, t) \in [\Omega - A] \times [0, T]$.

Proof. Define

$$M_n(\omega) = \sup_{t \in [0,T]} [B_n(\omega, t) - B(\omega, t)]. \tag{37}$$

Due to the sample continuity of $B_n(\omega, t)$ and $B(\omega, t)$, for each n this supremum (maximum) is finite with probability one, and it can be established by means of the values of t of a countable subset of $[0, T]$, dense in it. Hence, $M_n(\omega)$ is a finite Borel function defined on the $\mathcal{A}$-measurable sections of $[B_n(\omega, t) - B(\omega, t)]$ at values of $t \in [0, T]$. It is thus $\mathcal{A}$-measurable.

At each fixed ω, the trajectories converge uniformly on $[0, T]$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} M_n(\omega) = 0$ almost surely. Applying the Theorem of Egorov on the sequence $\{M_n(\omega)\}$, it follows that there is a set $A \in \mathcal{A}$, $P(A) < \epsilon$, such that $M_n(\omega) \to 0$ as $n \to \infty$ uniformly in $\omega \in \Omega - A$. Hence, $B_n(\omega, t) \to B(\omega, t)$ as $n \to \infty$ uniformly in $(\omega, t) \in [\Omega - A] \times [0, T]$.

(C) *The Convergence of $\xi_n(t)$ to $\xi(t)$.*

According to Theorem 3 and the discussion at the beginning of this section, and recalling the notation convention, we have

$$\begin{aligned}\xi_n(t) &= \overline{\int_0^t} g(t, \tau)\, d\beta_n(\tau) = \int_0^t g(t, \tau)\, d\beta_n(\tau) \\ &= -\int_0^t \beta_n(\tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau\end{aligned} \tag{38}$$

with representation

$$X_n(\omega, t) = -\int_0^t B_n(\omega, \tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau \qquad \text{on} \qquad \Omega \times [0, T], \tag{39}$$

and

$$\xi(t) = \overline{\int_0^t} g(t, \tau)\, d\beta(\tau) = -\int_0^t \beta(\tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau \tag{40}$$

with representation

$$X(\omega, t) = -\int_0^t B(\omega, t) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau \qquad \text{on} \qquad \Omega \times [0, T]. \tag{41}$$

All resulting integrals given above are stochastic sample integrals as well as mean square integrals.

THEOREM 6. *The sequence $\{\xi_n(t)\}$ converges to $\xi(t)$ in mean square, uniformly in $t \in [0, T]$.*

Proof. We need to show that

$$\overline{\int_0^t} \beta_n(\tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau \to \overline{\int_0^t} \beta(\tau) \frac{\partial g(t, \tau)}{\partial \tau}\, d\tau \qquad \text{as} \qquad n \to \infty \tag{42}$$

uniformly in $t \in [0, T]$. This is a well known result in the analysis in mean square (or in Banach spaces), since, according to Theorem 4, $\beta_n(t) \to \beta(t)$ in mean square as $n \to \infty$, uniformly in $t \in [0, T]$. (This proof could also be established with the aid of the covariance functions).

It is analogously proved that, at each ω, the trajectories of $X_n(\omega, t)$, $n = 1, 2, \ldots$, converge to $X_\omega(t)$, uniformly in $t \in [0, T]$ as $n \to \infty$.

Following the proof of Theorem 5 we can show

THEOREM 7. *Given $\epsilon > 0$, there is a set $A \in \mathcal{A}$ with $P(A) < \epsilon$ such that $X_n(\omega, t) \to X(\omega, t)$ as $n \to \infty$, uniformly in $(\omega, t) \in [\Omega - A] \times [0, T]$.*

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